BACK-PAPER   ISI BANGALORE   JULY 2016

B MATH ALGEBRA II   100 POINTS

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(c) By default, F denotes a field.

---

1. [7 points] Prove that a linear system $A\mathbf{x} = \mathbf{b}$ has a solution iff the rank of the appended matrix $A|\mathbf{b}$ is the same as the rank of A.

2. [27 points] For the matrix A given below, find the characteristic polynomial, eigenvalues and eigenvectors (over $\mathbb{C}$ if necessary). Find an invertible 3×3 matrix X such that XAX^{-1} is a diagonal matrix. Using this, compute A^{100}.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

3. [14 points] Give an example of 2 square matrices A, B of the same size such that the characteristic polynomials of A, B are equal but their minimal polynomials are not equal. Justify your answer.

4. [14 points] Suppose A is an square matrix such that $A^2 = A$. Prove that A is similar to a diagonal matrix having diagonal entries either 1 or 0.

5. [14 points] Find an orthonormal basis of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$[1,1,1,0,0]^t, \quad [0,0,1,1,1]^t, \quad [1,1,2,1,1]^t, \quad [0,0,1,0,0].$$

6. [10 points] Classify all 3×3 symmetric matrices over $\mathbb{C}$ up to bilinear equivalence, i.e., for the equivalence relation given by $A \sim B$ whenever $B = X^tAX$ for some invertible 3×3 matrix over $\mathbb{C}$, list all the equivalence classes for "$\sim$".

7. [14 points] Supply proofs of the following statements proved in class.
   (i) Any eigenvalue λ of a unitary matrix satisfies $\lambda\bar{\lambda} = 1$.
   (ii) For a Hermitian symmetric matrix A, if λ_1, λ_2 are distinct eigenvalues of A and v_1, v_2 are eigenvectors corresponding to these eigenvalues, then v_1, v_2 are orthogonal.